

Consistency relations for large scale structures with primordial non-Gaussianities

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We investigate how the consistency relations of large-scale structures are modified when the initial density field is not Gaussian. We consider both scenarios where the primordial density field can be written as a nonlinear functional of a Gaussian field and more general scenarios where the probability distribution of the primordial density field can be expanded around the Gaussian distribution, up to all orders over δ_{L0} . Working at linear order over the non-Gaussianity parameters $f_{NL}^{(n)}$ or S_n , we find that the consistency relations for the matter density fields are modified as they include additional contributions that involve all-order mixed linear-nonlinear correlations $\langle \prod \delta_L \prod \delta \rangle$. We derive the conditions needed to recover the simple Gaussian form of the consistency relations. This corresponds to scenarios that become Gaussian in the squeezed limit. Our results also apply to biased tracers, and velocity or momentum cross-correlations.

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I. INTRODUCTION

Large-scale matter inhomogeneities in the Universe, as partly probed by luminous distributions such as galaxies and clusters, are thought to have evolved from tiny density fluctuations under the influence of both gravity and cosmic expansion. Since the observations of large-scale structures are basically made with measurements of both redshift and angular positions, the three-dimensional matter distribution contains valuable cosmological information, complementary to the cosmic microwave background observed in a two-dimensional sky. With the precision measurements provided by the ongoing and future surveys, the large-scale structure observations may also give us hints and clues beyond the standard cosmological model.

While the evolution of matter inhomogeneities until the time of decoupling is fully described by the linear theory, based on the cosmological Boltzmann equations coupled with Einstein's theory of general relativity (e.g., [1]), the development of gravitational clustering becomes significant at late time, and it eventually enters the nonlinear stage, where the applicability of analytic predictions is severely restricted and only partly accessible through perturbation theory [2]. In general, numerical simulations are required for quantitative cosmological studies.

However, it has been recently recognized that there exist nonperturbative statistical relationships that hold even in the nonlinear regime. These so-called *consistency relations* represent a nontrivial coupling between large- and small-scale modes, and lead to exact relationships between higher- and lower-order statistics of density fluctuations (e.g., [3–11]). They generically hold not only for the density fluctuations of the matter distribution but

also for biased tracers. The original consistency relations are kinematic and vanish at equal times at leading order. One can also take into account next-to-leading order contributions to obtain relations that remain nonzero at equal times [12–14], but this requires some additional approximations so that these relations are no longer exact. On the other hand, by considering cross-correlations between density and velocity, or momentum, fields, one can again derive exact relations, analogs of the original density relations, that remain nonzero at equal times [15] and are potentially important as they may be directly measurable via observations. Since these exact consistency relations only rely on the weak equivalence principle and the Gaussianity of the primordial fluctuations, their measurement could be a powerful test of the fundamental hypothesis of the standard cosmological model.

A few works have considered how these relations are modified when one of the underlying assumptions is not fulfilled, namely if we have violations of the weak equivalence principle in modified-gravity scenarios [5, 7, 16], or primordial non-Gaussianities [4], by studying a few examples. This aspect is crucial if we wish to test the standard cosmological model through the measurement of these consistency relations. In this paper, we investigate in more details how the consistency relations are modified in the presence of primordial non-Gaussianity, considering both very general non-Gaussian models and all-order consistency relations.

The primordial non-Gaussianity is now severely constrained through the measurement of cosmic microwave background anisotropies [17, 18], but the outcome of these tight constraints relies on several model-dependent assumptions. This is one of the reasons why the large-scale structure observations still attract attention as an independent/complementary probe of primordial non-

Gaussianity. A particularly remarkable feature that has been recently recognized is a strong enhancement of the halo/galaxy clustering bias on large scales in the presence of the so-called local-type non-Gaussianity (e.g., [19–21]). This enhancement indeed arises from a tight coupling between the large- and small-scale modes through the squeezed limit of higher-order matter correlations (e.g., [22]), which is exactly the case we are looking at in the consistency relations. It is thus interesting to see how the structure of the correlation hierarchy is generically modified when the initial fluctuations are not Gaussian.

Hereafter, we consider non-Gaussian primordial matter fluctuations described by their Taylor expansion over a Gaussian field, or by a probability distribution that can be expanded around the Gaussian. We derive rather generic expressions for the higher-order correlations of matter fluctuations that remain valid in the nonlinear regime. We also consider biased tracers. Our results show that in the non-Gaussian case the consistency relations are largely modified by additional contributions involving all-order mixed linear-nonlinear correlations. Based on this, we discuss the necessary conditions to recover the usual consistency relations that hold for Gaussian initial conditions, and see how these conditions are satisfied or violated in specific non-Gaussian models.

This paper is organized as follows. In Sec. II, we begin by describing non-Gaussian primordial density fields as Taylor expansions of an auxiliary Gaussian field. Next, we consider the more general case where the primordial density field is merely defined by its non-Gaussian probability distribution, which we assume can be expanded around the Gaussian. We also recall several popular non-Gaussian models that provide useful examples. Sec. III considers the basis to derive the consistency relations [9], and derives the relation between the response functions of cosmic density fields, with respect to the linear density field, and higher-order correlation functions. Sec. IV then presents our main results, which describe how the consistency relations of density fields are generically modified in the presence of primordial non-Gaussianity. Specific results for several non-Gaussian models are also given. Further, Sec. V discusses the consistency relations for velocity and momentum fields. Finally, Sec. VI is devoted to the conclusion and summary of the results.

II. MODELS OF PRIMORDIAL NON-GAUSSIANITIES

In this section, as a starting point to derive the correlation hierarchy in the presence of primordial non-Gaussianity, we give a general framework to deal with non-Gaussian matter fluctuations at linear order. In Sec. II A, we present a description of non-Gaussian primordial density fields through their Taylor expansion over auxiliary Gaussian fields. In Sec. II B, this is generalized to the probability distribution functional for non-Gaussian primordial fluctuations, which is later used to

derive the consistency relations for matter fluctuations. As a simple and illustrative example, in Sec. II C, we consider the non-Gaussian model in which the Taylor expansion is truncated at second order. Specific models to realize such a non-Gaussianity are described in Sec. II D, and we briefly discuss their distinct features in the squeezed limit.

A. Primordial density field as a nonlinear functional of a Gaussian field

Simple models of primordial non-Gaussianities can be built where the primordial (i.e. linear) density contrast $\delta_L(\mathbf{x}, \tau)$ can be written as a nonlinear functional of a Gaussian field $\chi(\mathbf{x}, \tau)$. Linearizing over the non-Gaussianity parameters $f_{\text{NL}}^{(n)}$, as we do throughout this study, we write in Fourier space

$$\delta_{L0}(\mathbf{k}) = \chi_0(\mathbf{k}) + \sum_{n=2}^{\infty} \int \prod_{i=1}^n d\mathbf{k}_i \delta_D\left(\mathbf{k} - \sum_{i=1}^n \mathbf{k}_i\right) \times f_{\text{NL}0}^{(n)}(\mathbf{k}_1, \dots, \mathbf{k}_n) \prod_{i=1}^n \chi_0(\mathbf{k}_i), \quad (1)$$

where the subscript “0” denotes that we normalize the fields today at $z = 0$ and we can take the kernels $f_{\text{NL}}^{(n)}$ to be symmetric. Throughout this paper we assume statistical homogeneity, hence the Dirac factors in Eq.(1), and isotropy, which yields the constraint

$$f_{\text{NL}}^{(n)}(\mathbf{k}_1, \dots, \mathbf{k}_n) = f_{\text{NL}}^{(n)}(-\mathbf{k}_1, \dots, -\mathbf{k}_n). \quad (2)$$

For the sake of generality we keep track of all orders $n \geq 2$ in the nonlinear functional $\delta_L[\chi]$, but in practice one often only includes the quadratic or cubic terms, which are then denoted as $f_{\text{NL}} = f_{\text{NL}}^{(2)}$ and $g_{\text{NL}} = f_{\text{NL}}^{(3)}$. The fields δ_L , χ , and the kernels $f_{\text{NL}}^{(n)}$ evolve with redshift as

$$\delta_L = D_+ \delta_{L0}, \quad \chi = D_+ \chi_0, \quad f_{\text{NL}}^{(n)} = D_+^{1-n} f_{\text{NL}0}^{(n)}, \quad (3)$$

where $D_+(\tau)$ is the linear growing mode. The kernels $f_{\text{NL}}^{(n)}$ must satisfy the constraint

$$n \geq 2 : \quad f_{\text{NL}}^{(n)}(\mathbf{k}_1, \dots, \mathbf{k}_n) = 0 \quad \text{for} \quad \mathbf{k}_1 + \dots + \mathbf{k}_n = 0, \quad (4)$$

so that $\delta_L(0) = 0$. At linear order over $f_{\text{NL}}^{(n)}$ this yields the primordial power spectrum

$$\begin{aligned} \langle \delta_{L0}(\mathbf{k}) \delta_{L0}(-\mathbf{k}) \rangle' &\equiv P_{L0}(k) \\ &= P_{\chi_0}(k) + 2P_{\chi_0}(k) \sum_{n=1}^{\infty} (2n+1)!! \int \prod_{i=1}^n d\mathbf{k}'_i \\ &\times f_{\text{NL}0}^{(2n+1)}(\mathbf{k}, \mathbf{k}'_1, -\mathbf{k}'_1, \dots, \mathbf{k}'_n, -\mathbf{k}'_n) \prod_{i=1}^n P_{\chi_0}(k'_i) \end{aligned} \quad (5)$$

and the primordial bispectrum

$$\begin{aligned} \langle \delta_{L0}(\mathbf{k}_1) \delta_{L0}(\mathbf{k}_2) \delta_{L0}(\mathbf{k}_3) \rangle' &= P_{\chi_0}(k_2) P_{\chi_0}(k_3) \sum_{n=1}^{\infty} 2n \\ &\times (2n-1)!! \int \prod_{i=1}^{n-1} d\mathbf{k}'_i \prod_{i=1}^{n-1} P_{\chi_0}(k'_i) \\ &\times f_{\text{NL0}}^{(2n)}(\mathbf{k}_2, \mathbf{k}_3, \mathbf{k}'_1, -\mathbf{k}'_1, \dots, \mathbf{k}'_{n-1}, -\mathbf{k}'_{n-1}) + 2 \text{ cyc.} \end{aligned} \quad (6)$$

where the prime in $\langle \dots \rangle'$ denotes that we removed the Dirac factors $\delta_D(\sum \mathbf{k}_i)$.

Because the relevant field for the formation of large-scale structures is the linear density field δ_{L0} rather than the auxiliary Gaussian field χ_0 , it is convenient to eliminate χ_0 in favor of δ_{L0} . This is possible because at linear order over $f_{\text{NL}}^{(n)}$ we can invert Eq.(1) as

$$\begin{aligned} \chi_0(\mathbf{k}) &= \delta_{L0}(\mathbf{k}) - \sum_{n=2}^{\infty} \int \prod_{i=1}^n d\mathbf{k}_i \delta_D\left(\mathbf{k} - \sum_{i=1}^n \mathbf{k}_i\right) \\ &\times f_{\text{NL0}}^{(n)}(\mathbf{k}_1, \dots, \mathbf{k}_n) \prod_{i=1}^n \delta_{L0}(\mathbf{k}_i) + \mathcal{O}(f_{\text{NL}}^2). \end{aligned} \quad (7)$$

The generating functional $\langle e^{j \cdot \delta_{L0}} \rangle$ reads as

$$\langle e^{j \cdot \delta_{L0}} \rangle = \int \mathcal{D}\chi_0 e^{j \cdot \delta_{L0}[\chi_0]} e^{-\chi_0 \cdot C_{\chi_0}^{-1} \cdot \chi_0/2}, \quad (8)$$

as χ_0 is Gaussian and we introduced the inverse matrix $C_{\chi_0}^{-1}$ of the two-point correlation $C_{\chi_0}(\mathbf{k}_1, \mathbf{k}_2) \equiv \delta_D(\mathbf{k}_1 + \mathbf{k}_2) P_{\chi_0}(k_1)$. Using Eq.(7) we can change variable to δ_{L0} to obtain

$$\begin{aligned} \langle e^{j \cdot \delta_{L0}} \rangle &= \int \mathcal{D}\delta_{L0} J \exp \left[j \cdot \delta_{L0} - (\delta_{L0} - \sum_n f_{\text{NL0}}^{(n)} \delta_{L0} \dots \delta_{L0}) \right. \\ &\quad \left. \cdot C_{\chi_0}^{-1} \cdot (\delta_{L0} - \sum_n f_{\text{NL0}}^{(n)} \delta_{L0} \dots \delta_{L0})/2 \right], \end{aligned} \quad (9)$$

where the Jacobian determinant reads at linear order over $f_{\text{NL}}^{(n)}$ as

$$\begin{aligned} J &\equiv \left| \det \left(\frac{\mathcal{D}\chi_0}{\mathcal{D}\delta_{L0}} \right) \right| \\ &= 1 - \sum_{n=3}^{\infty} n \int d\mathbf{k} \int \prod_{i=1}^{n-1} d\mathbf{k}'_i \delta_D\left(\sum_{i=1}^{n-1} \mathbf{k}'_i\right) \\ &\quad \times f_{\text{NL0}}^{(n)}(\mathbf{k}, \mathbf{k}'_1, \dots, \mathbf{k}'_{n-1}) \prod_{i=1}^{n-1} \delta_{L0}(\mathbf{k}'_i). \end{aligned} \quad (10)$$

Here we used the fact that the first term $n=2$ in Eq.(10) vanishes because $\delta_{L0}(\mathbf{k}'_1=0)=0$ [as enforced by the Gaussian weight with $P_{\chi_0}(0)=0$]. Expanding Eq.(9) up to first order over $f_{\text{NL}}^{(n)}$ we obtain

$$\begin{aligned} \langle e^{j \cdot \delta_{L0}} \rangle &= \int \mathcal{D}\delta_{L0} e^{j \cdot \delta_{L0} - \delta_{L0} \cdot C_{\chi_0}^{-1} \cdot \delta_{L0}/2} \left[1 + \sum_{n=2}^{\infty} \int \prod_{i=1}^n d\mathbf{k}_i \right. \\ &\quad \left. \times \delta_D\left(\sum_{i=1}^n \mathbf{k}_i\right) S_n(\mathbf{k}_1, \dots, \mathbf{k}_n) \prod_{i=1}^n \delta_{L0}(\mathbf{k}_i) \right] \end{aligned} \quad (11)$$

where we introduced the symmetric kernels, for $n \geq 2$,

$$\begin{aligned} S_n(\mathbf{k}_1, \dots, \mathbf{k}_n) &= -(n+1) \int d\mathbf{k}' f_{\text{NL0}}^{(n+1)}(\mathbf{k}', \mathbf{k}_1, \dots, \mathbf{k}_n) \\ &\quad + \frac{1}{n} \sum_{\text{cyc.}} \frac{1}{P_{\chi_0}(k_1)} f_{\text{NL0}}^{(n-1)}(\mathbf{k}_2, \dots, \mathbf{k}_n), \end{aligned} \quad (12)$$

with $f_{\text{NL0}}^{(1)}(k) \equiv 0$ and the sum runs over the n cyclic permutations of $\{\mathbf{k}_1, \dots, \mathbf{k}_n\}$. This means that the probability distribution functional of the linear density field δ_{L0} reads as

$$\begin{aligned} \mathcal{P}(\delta_{L0}) &= e^{-\int d\mathbf{k} \delta_{L0}(\mathbf{k}) \delta_{L0}(-\mathbf{k})/2 P_{\chi_0}(k)} \left[1 + \sum_{n=2}^{\infty} \int \prod_{i=1}^n d\mathbf{k}_i \right. \\ &\quad \left. \times \delta_D\left(\sum_{i=1}^n \mathbf{k}_i\right) S_n(\mathbf{k}_1, \dots, \mathbf{k}_n) \prod_{i=1}^n \delta_{L0}(\mathbf{k}_i) \right]. \end{aligned} \quad (13)$$

B. Primordial density field with a non-Gaussian probability distribution functional

Independently of any nonlinear mapping to an auxiliary Gaussian field χ_0 , as in Eq.(1), we can define the initial conditions of the cosmological density field δ_{L0} by its probability distribution functional $\mathcal{P}(\delta_{L0})$. When we go beyond the Gaussian case we face an infinite number of possibilities, however we may consider distributions of the same form as Eq.(13). Here the kernels S_n are no longer given in terms of kernels $f_{\text{NL0}}^{(n)}$ as in Eq.(12). They define the probability distribution (13) of the initial conditions, through the expansion of its non-Gaussian part over δ_{L0} . This approach is more general than the explicit models (1) and it also applies to multi-field scenarios, where the final density perturbations are produced by a combination of several primordial fields.

The term $n=1$ in the expansion (13) still vanishes because $\delta_{L0}(0)=0$. The normalization condition $\langle 1 \rangle = 1$ implies that the kernels S_n must obey the constraint

$$\begin{aligned} \sum_{n=1}^{\infty} (2n-1)!! \int \prod_{i=1}^n d\mathbf{k}_i S_{2n}(\mathbf{k}_1, -\mathbf{k}_1, \dots, \mathbf{k}_n, -\mathbf{k}_n) \\ \times \prod_{i=1}^n P_{\chi_0}(k_i) = 0, \end{aligned} \quad (14)$$

while the condition $\langle \delta_{L0}(\mathbf{k}) \rangle = 0$ gives the constraint

$$\begin{aligned} \sum_{n=1}^{\infty} (2n+1)!! \int \prod_{i=1}^n d\mathbf{k}_i S_{2n+1}(0, \mathbf{k}_1, -\mathbf{k}_1, \dots, \mathbf{k}_n, -\mathbf{k}_n) \\ \times P_{\chi_0}(0) \prod_{i=1}^n P_{\chi_0}(k_i) = 0. \end{aligned} \quad (15)$$

We can check that the explicit models (12) satisfy the conditions (14)-(15). The first condition (14) is satisfied because of the cancellation in the sum over

n between the two terms in Eq.(12). The second condition (15) is satisfied because $P_{\chi_0}(0) = 0$ and $f_{\text{NL0}}^{(2n)}(\mathbf{k}_1, -\mathbf{k}_1, \dots, \mathbf{k}_n, -\mathbf{k}_n) = 0$ from Eq.(4). As expected, it involves the constraints on P_{χ_0} and $f_{\text{NL0}}^{(n)}$ found in section II A that are associated with the condition $\delta_L(0) = 0$. [The property $P_{\chi_0}(0) = 0$ is not sufficient to ensure Eq.(15) because of a factor $1/P_{\chi_0}(0)$ that arises from the second term in Eq.(12).]

Then, the linear density power spectrum reads as

$$P_{L0}(k) \equiv \langle \delta_{L0}(\mathbf{k}) \delta_{L0}(-\mathbf{k}) \rangle' = P_{\chi_0}(k) + P_{\chi_0}(k)^2 \times \sum_{n=1}^{\infty} 2n(2n-1)!! \int \prod_{i=1}^{n-1} d\mathbf{k}'_i \prod_{i=1}^{n-1} P_{\chi_0}(k'_i) \times S_{2n}(\mathbf{k}, -\mathbf{k}, \mathbf{k}'_1, -\mathbf{k}'_1, \dots, \mathbf{k}'_{n-1}, -\mathbf{k}'_{n-1}), \quad (16)$$

while the primordial density bispectrum reads as

$$\langle \delta_{L0}(\mathbf{k}_1) \delta_{L0}(\mathbf{k}_2) \delta_{L0}(\mathbf{k}_3) \rangle' = P_{\chi_0}(k_1) P_{\chi_0}(k_2) P_{\chi_0}(k_3) \times \sum_{n=1}^{\infty} 2n(2n+1)!! \int \prod_{i=1}^{n-1} d\mathbf{k}'_i \prod_{i=1}^{n-1} P_{\chi_0}(k'_i) \times S_{2n+1}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}'_1, -\mathbf{k}'_1, \dots, \mathbf{k}'_{n-1}, -\mathbf{k}'_{n-1}). \quad (17)$$

In the case of the generalized local models (1) we can check from Eq.(12) that we recover the expressions (5) and (6).

C. S_3 -type primordial non-Gaussianity

As a simple example of non-Gaussian models, let us consider the case of quadratic models, where the expansion (1) stops at the quadratic term,

$$\delta_{L0}(\mathbf{k}) = \chi_0(\mathbf{k}) + \int d\mathbf{k}_1 d\mathbf{k}_2 \delta_D(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) \times f_{\text{NL0}}(\mathbf{k}_1, \mathbf{k}_2) \chi_0(\mathbf{k}_1) \chi_0(\mathbf{k}_2), \quad (18)$$

that is,

$$f_{\text{NL}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2) = f_{\text{NL}}(\mathbf{k}_1, \mathbf{k}_2), \quad f_{\text{NL}}^{(n)} = 0 \text{ for } n \geq 3. \quad (19)$$

This includes for instance the local model described in Eqs.(27)-(29) below. Then, the kernels S_n introduced in Eq.(12) are

$$S_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{f_{\text{NL0}}(\mathbf{k}_2, \mathbf{k}_3)}{3P_{\chi_0}(k_1)} + 2 \text{ cyc.} \quad (20)$$

and

$$S_n = 0 \text{ for } n \neq 3. \quad (21)$$

The normalization constraint (14) is trivially satisfied as it reads $0 = 0$. The condition (15) associated with the constraint $\langle \delta_{L0}(\mathbf{k}) \rangle = 0$ writes as

$$\int d\mathbf{k} S_3(0, \mathbf{k}, -\mathbf{k}) P_{\chi_0}(0) P_{\chi_0}(k) = 0. \quad (22)$$

From Eq.(20) this reads as

$$\int d\mathbf{k} f_{\text{NL0}}(\mathbf{k}, -\mathbf{k}) P_{\chi_0}(k) + 2 \int d\mathbf{k} f_{\text{NL0}}(0, \mathbf{k}) P_{\chi_0}(0) = 0, \quad (23)$$

which is satisfied as both terms vanish, thanks to Eq.(4) and $P_{\chi_0}(0) = 0$.

The power spectrum (16) is simply

$$P_{L0}(k) = P_{\chi_0}(k), \quad (24)$$

while the bispectrum (17) writes as

$$\langle \delta_{L0}(\mathbf{k}_1) \delta_{L0}(\mathbf{k}_2) \delta_{L0}(\mathbf{k}_3) \rangle' = P_{\chi_0}(k_1) P_{\chi_0}(k_2) P_{\chi_0}(k_3) \times 6 S_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3). \quad (25)$$

Using Eq.(20) we recover the usual result

$$\langle \delta_{L0}(\mathbf{k}_1) \delta_{L0}(\mathbf{k}_2) \delta_{L0}(\mathbf{k}_3) \rangle' = 2f_{\text{NL0}}(\mathbf{k}_2, \mathbf{k}_3) \times P_{\chi_0}(k_2) P_{\chi_0}(k_3) + 2 \text{ cyc.} \quad (26)$$

More generally, beyond the explicit quadratic models (18), we can define non-Gaussian models by the kernel S_3 itself, without reference to an auxiliary Gaussian field χ_0 . With $S_n = 0$ for $n \neq 3$, this provides the simplest probability distribution $\mathcal{P}(\delta_{L0})$ of Eq.(13), which defines the initial conditions for the density field, that is fully determined by the power spectrum and the bispectrum. Therefore, this can be seen as the simplest model (in terms of the distribution function) of a non-Gaussian primordial density field when we only know its second and third-order moments. Then, the kernel S_3 simply needs to satisfy the condition (22) to provide a physical model.

D. Explicit examples of non-Gaussian models

1. Local model

A specific example of such S_3 -type primordial non-Gaussianity is the quadratic local model, where we write Bardeen's potential Φ as

$$\text{local type: } \Phi(\mathbf{x}) = \phi(\mathbf{x}) + f_{\text{NL}} (\phi(\mathbf{x})^2 - \langle \phi^2 \rangle), \quad (27)$$

where ϕ is a Gaussian field and f_{NL} a parameter. On subhorizon scales the Poisson equation gives

$$\delta_L(\mathbf{k}, \tau) = \alpha(k, \tau) \Phi(\mathbf{k}) \text{ with } \alpha(k, \tau) = \frac{2c^2 k^2 T(k) D_+(\tau)}{3\Omega_{m0} H_0^2}, \quad (28)$$

where $T(k)$ is the transfer function. Then, defining $\chi(\mathbf{k}, \tau) = \alpha(k, \tau) \phi(\mathbf{k})$, we obtain the relation (18) with

$$\text{local type: } f_{\text{NL}}(\mathbf{k}_1, \mathbf{k}_2; \tau) = f_{\text{NL}} \frac{\alpha(\mathbf{k}_1 + \mathbf{k}_2, \tau)}{\alpha(k_1, \tau) \alpha(k_2, \tau)}. \quad (29)$$

The general form (1) can describe a more general scale dependence of the primordial non-Gaussianity than the specific kernel (29) and higher-order contributions.

2. Equilateral and orthogonal models

Other than the local-type primordial non-Gaussianity, there are models discussed in the literature that produce distinctive features in the primordial bispectrum. The so-called equilateral- and orthogonal-type non-Gaussianities are known to characterize a possible deviation from the simplest single-field inflation scenario (e.g., [23, 24]). While the primordial bispectrum of the former type has peaks at equilateral configurations, the latter type gives a signal peaked both on equilateral and flat-triangle configurations but with opposite sign. The bispectrum of these models can be expressed by the template [25]

$$\begin{aligned} \langle \Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\Phi(\mathbf{k}_3) \rangle' &= 6 f_{\text{NL}} \\ &\times \left\{ c_1 [P_\Phi(k_1)P_\Phi(k_2) + 2 \text{ cyc.}] \right. \\ &+ c_2 [P_\Phi(k_1)^{1/3}P_\Phi(k_2)^{2/3}P_\Phi(k_3) + 5 \text{ cyc.}] \\ &\left. + c_3 [P_\Phi(k_1)P_\Phi(k_2)P_\Phi(k_3)]^{2/3} \right\} \end{aligned} \quad (30)$$

with the coefficients being $(c_1, c_2, c_3) = (-1, 1, -2)$ for equilateral type and $(-3, 3, -8)$ for orthogonal type. Note that with the expression given above, the local model (27) corresponds to $(c_1, c_2, c_3) = (1/3, 0, 0)$. Using the relation (28) between the primordial gravitational potential Φ and the matter-era linear density contrast δ_L , Eq.(30) gives the linear matter density bispectrum

$$\begin{aligned} \langle \delta_L(\mathbf{k}_1)\delta_L(\mathbf{k}_2)\delta_L(\mathbf{k}_3) \rangle' &= 6 f_{\text{NL}} \\ &\times \left\{ c_1 \left[\frac{\alpha(k_3)P_L(k_1)P_L(k_2)}{\alpha(k_1)\alpha(k_2)} + 2 \text{ cyc.} \right] \right. \\ &+ c_2 \left[\frac{\alpha(k_1)^{1/3}P_L(k_1)^{1/3}P_L(k_2)^{2/3}P_L(k_3)}{\alpha(k_2)^{1/3}\alpha(k_3)} + 5 \text{ cyc.} \right] \\ &\left. + c_3 \frac{[P_L(k_1)P_L(k_2)P_L(k_3)]^{2/3}}{[\alpha(k_1)\alpha(k_2)\alpha(k_3)]^{1/3}} \right\}. \end{aligned} \quad (31)$$

Although the equilateral and orthogonal models are primarily defined via the shape of the bispectrum, they can also be characterized by the quadratic model (18). From Eq.(26) we can check that we recover the bispectrum (31) with the choice

$$\begin{aligned} f_{\text{NL}}(\mathbf{k}_1, \mathbf{k}_2) &= f_{\text{NL}} \left\{ 3c_1 \frac{\alpha(k_3)}{\alpha(k_1)\alpha(k_2)} \right. \\ &+ 3c_2 \left[\frac{\alpha(k_3)^{1/3}P_L(k_3)^{1/3}}{\alpha(k_1)\alpha(k_2)^{1/3}P_L(k_2)^{1/3}} + \frac{\alpha(k_3)^{1/3}P_L(k_3)^{1/3}}{\alpha(k_2)\alpha(k_1)^{1/3}P_L(k_1)^{1/3}} \right] \\ &\left. + c_3 \frac{P_L(k_3)^{2/3}}{[\alpha(k_1)\alpha(k_2)\alpha(k_3)P_L(k_1)P_L(k_2)]^{1/3}} \right\}, \end{aligned} \quad (32)$$

where $\mathbf{k}_3 = -\mathbf{k}_1 - \mathbf{k}_2$.

In terms of the kernel S_3 we obtain from Eqs.(25) and

(31) the expression

$$\begin{aligned} S_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) &= f_{\text{NL}} \left\{ c_1 \left[\frac{\alpha(k_3)}{\alpha(k_1)\alpha(k_2)P_L(k_3)} + 2 \text{ cyc.} \right] \right. \\ &+ c_2 \left[\frac{\alpha(k_1)^{1/3}}{\alpha(k_2)^{1/3}\alpha(k_3)P_L(k_1)^{2/3}P_L(k_2)^{1/3}} + 5 \text{ cyc.} \right] \\ &\left. + c_3 \frac{1}{[\alpha(k_1)\alpha(k_2)\alpha(k_3)P_L(k_1)P_L(k_2)P_L(k_3)]^{1/3}} \right\}. \end{aligned} \quad (33)$$

3. Squeezed limit

In the squeezed limit where one wavenumber goes to zero, the kernel S_3 of the local, equilateral and orthogonal models defined by Eq.(33) behaves as

$$\begin{aligned} S_3(\mathbf{k}', \mathbf{k}, -\mathbf{k})_{k' \rightarrow 0} &\sim f_{\text{NL}} \left[\frac{2(c_1 + c_2)}{\alpha(k')P_L(k)} \right. \\ &\left. + \frac{c_3}{\alpha(k')^{1/3}\alpha(k)^{2/3}P_L(k')^{1/3}P_L(k)^{2/3}} \right]. \end{aligned} \quad (34)$$

For a large-scale power spectrum index n_s , $P_L(k) \propto k^{n_s}$, this gives

$$k' \rightarrow 0 : \quad P_L(k')S_3(\mathbf{k}', \mathbf{k}, -\mathbf{k}) \sim f_{\text{NL}} [(c_1 + c_2)k'^{n_s-2} + c_3k'^{2(n_s-1)/3}], \quad (35)$$

where we did not write the k -dependent factors. Since $n_s \simeq 0.96$, we can see that both terms diverge at low k' . The divergence is strongest for the local model where $c_1 + c_2 = 1/3$, whereas it is rather mild for the equilateral and orthogonal models where $c_1 + c_2 = 0$. As noted in Refs. [24, 25], for the orthogonal model the template (30) is not very accurate in the squeezed limit and it gives a spurious divergence. If one wishes to study the orthogonal models physically defined from the inflationary models one needs to use more intricate templates that regularize the squeezed limit [24, 25]. On the other hand, the template in Eq. (30) may be used as a phenomenological model, independently of a specific theoretical scenario.

For models such as the local model (29), where the expression (35) diverges or remains nonzero, we violate the condition (23) associated with the requirement $\langle \delta_L \rangle = 0$. Indeed, to satisfy this constraint the expression (35) should vanish for $k' \rightarrow 0$. In practice, for instance in numerical simulations, such infrared divergences are regularized by the finite size of the system. This means that there is no power on very large scales, beyond the simulation box, and $P_L(k')S_3(\mathbf{k}', \mathbf{k}, -\mathbf{k})$ goes to zero at low k' . However, these models show large non-Gaussianities in the squeezed limit, on large cosmological scales below the cutoff, and they lead to different behaviors than the models where $P_L(k')S_3(\mathbf{k}', \mathbf{k}, -\mathbf{k})$ smoothly goes to zero without introducing an infrared cutoff.

III. CORRELATION AND RESPONSE FUNCTIONS WITH PRIMORDIAL NON-GAUSSIANITIES

Although the linear density field δ_{L0} is not Gaussian, we can apply with only modest modifications the approach that was described in [9, 12] to obtain consistency relations for cosmological structures. This relies on relations between correlation and response functions. Thus, let us consider a set of quantities $\{\rho_i\}$ that are functionals of δ_{L0} , typically the nonlinear density contrasts $\{\delta(\mathbf{k}_i, \tau_i)\}$ at various wavenumbers and conformal times, and the mean response function $R(\mathbf{k}')$ of their product with respect to the field δ_{L0} (which defines the initial conditions of the system),

$$R^{1,m}(\mathbf{k}') = \left\langle \frac{\mathcal{D}[\rho_1 \dots \rho_m]}{\mathcal{D}\delta_{L0}(\mathbf{k}')} \right\rangle = \int \mathcal{D}\delta_{L0} \mathcal{P}(\delta_{L0}) \frac{\mathcal{D}[\rho_1 \dots \rho_m]}{\mathcal{D}\delta_{L0}(\mathbf{k}')} \quad (36)$$

Integrating by parts we obtain

$$R^{1,m}(\mathbf{k}') = - \int \mathcal{D}\delta_{L0} \frac{\mathcal{D}\mathcal{P}}{\mathcal{D}\delta_{L0}(\mathbf{k}')} \rho_1 \dots \rho_m. \quad (37)$$

Using the expression (13) this yields at linear order over the kernels S_n

$$\begin{aligned} R^{1,m}(\mathbf{k}') &= \langle \rho_1 \dots \rho_m \delta_{L0}(-\mathbf{k}') \rangle / P_{\chi_0}(k') - \sum_{n=2}^{\infty} n \int \prod_{i=1}^{n-1} d\mathbf{k}'_i \\ &\quad \times \delta_D(\mathbf{k}' + \mathbf{k}'_1 + \dots + \mathbf{k}'_{n-1}) S_n(\mathbf{k}', \mathbf{k}'_1, \dots, \mathbf{k}'_{n-1}) \\ &\quad \times \langle \rho_1 \dots \rho_m \prod_{i=1}^{n-1} \delta_{L0}(\mathbf{k}'_i) \rangle. \end{aligned} \quad (38)$$

Note that the statistical average is with respect to the non-Gaussian distribution (13), but for the second term with the factors S_n we can take the average with the Gaussian weight only, as we work at linear order over S_n or $f_{\text{NL}}^{(n)}$. Defining the mixed correlations $C^{\ell,m}(\mathbf{k}'_1, \dots, \mathbf{k}'_\ell)$ as

$$C^{\ell,m}(\mathbf{k}'_1, \dots, \mathbf{k}'_\ell) = \langle \delta_{L0}(\mathbf{k}'_1) \dots \delta_{L0}(\mathbf{k}'_\ell) \rho_1 \dots \rho_m \rangle, \quad (39)$$

Eq.(38) writes as

$$\begin{aligned} C^{1,m}(\mathbf{k}') &= P_{\chi_0}(k') R^{1,m}(-\mathbf{k}') + P_{\chi_0}(k') \sum_{n=2}^{\infty} n \int \prod_{i=1}^{n-1} d\mathbf{k}'_i \\ &\quad \times \delta_D\left(\mathbf{k}' - \sum_{i=1}^{n-1} \mathbf{k}'_i\right) S_n(-\mathbf{k}', \mathbf{k}'_1, \dots, \mathbf{k}'_{n-1}) \\ &\quad \times C^{n-1,m}(\mathbf{k}'_1, \dots, \mathbf{k}'_{n-1}). \end{aligned} \quad (40)$$

We recover the relation between correlation and response functions associated with Gaussian initial conditions if we set $S_n = 0$. Thus, the primordial non-Gaussianity introduces new terms that involve the higher-order mixed correlations $C^{n-1,m}$. Therefore, in the large-scale limit

$\mathbf{k}' \rightarrow 0$ we only recover the standard result if these new terms go to zero, that is,

$$k' \rightarrow 0 : \quad C^{1,m}(\mathbf{k}') - P_{\chi_0}(k') R^{1,m}(-\mathbf{k}') \rightarrow 0, \quad (41)$$

if we have

$$\begin{aligned} \text{“squeezed Gaussianity” : } &\text{for any } \mathbf{k}_1 + \dots + \mathbf{k}_n = 0, \\ &P_{\chi_0}(0) S_{n+1}(0, \mathbf{k}_1, \dots, \mathbf{k}_n) = 0. \end{aligned} \quad (42)$$

We can see that this condition is more stringent than the condition (15) associated with the constraint $\langle \delta_L \rangle = 0$. On the other hand, because it provides a natural way to satisfy Eq.(15), by making each term in that sum vanish, it defines a natural subclass of non-Gaussian models.

In particular, we can check that the generalized local models (12) typically belong to this class, as they obey Eq.(42) and this is the manner they satisfy Eq.(15). Indeed, as we have already seen below Eq.(15), $P_{\chi_0}(0) = 0$ and the term in S_n with a factor $1/P_{\chi_0}(0)$ that appears in Eq.(12) also vanishes because of the condition (4). However, as we have seen in section IID this is not the case of the explicit local, equilateral and orthogonal models, if they are defined by the template (30). This is because of the inverse powers of $\alpha(k)$ that appear in Eq.(33) and lead to infrared divergences. These divergences may be spurious and due to an inaccurate modeling of the squeezed limit [24]. In this case the condition (42) is satisfied, once we use a more accurate template, and we recover the Gaussian form (42). On the other hand, if the infrared divergence is meaningful, and only regularized by an additional cutoff at very large scales, on cosmological scales below this cutoff the relations (40) show significant deviations from the Gaussian form.

From the expression (17) we also find out that the property (42) implies that the primordial density bispectrum vanishes in the squeezed limit,

$$\langle \delta_{L0}(0) \delta_{L0}(\mathbf{k}) \delta_{L0}(-\mathbf{k}) \rangle' = 0. \quad (43)$$

This means that non-Gaussian models that have a non-zero squeezed bispectrum violate the relationship (41) between response and correlation functions. On the other hand, a large class of non-Gaussian models, such as the generalized local models (1), satisfy Eq.(42), which implies that they recover the relationship (41), which takes the same form as in the Gaussian case, and they have a vanishing squeezed bispectrum (43). From Eq.(16) we find that these models also satisfy

$$k \rightarrow 0 : \quad \frac{P_{L0}(k)}{P_{\chi_0}(k)} \rightarrow 1. \quad (44)$$

Thus, in a broad sense these models correspond to scenarios where the primordial density field becomes Gaussian on very large scales, $k \rightarrow 0$, or more precisely when at least one wavenumber goes to zero, in the squeezed limit. This is why we may call the property (42) as a squeezed Gaussianity criterion.

Thus, we find that for models where the primordial bispectrum vanishes in the squeezed limit (43) and non-Gaussianities are negligible in this limit, or more accurately where the condition (42) is satisfied, the correlation $C^{1,m}$ in the large-scale limit $k' \rightarrow 0$ is set by the gravitational dynamics (associated with the response $R^{1,m}$) as in the Gaussian case. Even though this relation takes the same form as in the Gaussian case, it goes beyond the Gaussian model as it includes an implicit dependence on the primordial non-Gaussianity, because both quantities $C^{1,m}$ and $R^{1,m}$ depend on the properties of the initial conditions of the system, hence on the kernels S_n .

IV. CONSISTENCY RELATIONS FOR THE DENSITY CONTRAST

A. General case at all orders

In practice, we consider the quantities $\{\rho_i\}$ to be non-linear fields, such as the nonlinear matter density contrasts $\delta(\mathbf{k}_i, \tau_i)$. As in the Gaussian case, they are fully determined by the linear field δ_{L0} that sets both the linear growing mode and the initial conditions (assuming as usual that decaying modes have had time to become negligible before gravitational clustering enters the nonlinear regime). Then, we consider the mixed matter density correlation and response functions

$$C_\delta^{1,m}(\mathbf{k}') = \langle \delta_{L0}(\mathbf{k}') \delta(\mathbf{k}_1, \tau_1) \dots \delta(\mathbf{k}_m, \tau_m) \rangle, \quad (45)$$

$$R_\delta^{1,m}(\mathbf{k}') = \left\langle \frac{\mathcal{D}[\prod_{j=1}^m \delta(\mathbf{k}_j, \tau_j)]}{\mathcal{D}\delta_{L0}(\mathbf{k}')} \right\rangle, \quad (46)$$

and the relationship (40) writes as

$$\begin{aligned} \langle \delta_{L0}(\mathbf{k}') \prod_{j=1}^m \delta(\mathbf{k}_j, \tau_j) \rangle &= P_{\chi_0}(k') \left\langle \frac{\mathcal{D}[\prod_{j=1}^m \delta(\mathbf{k}_j, \tau_j)]}{\mathcal{D}\delta_{L0}(-\mathbf{k}')} \right\rangle \\ &+ P_{\chi_0}(k') \sum_{n=2}^{\infty} n \int \prod_{i=1}^{n-1} d\mathbf{k}'_i \delta_D\left(\mathbf{k}' - \sum_{i=1}^{n-1} \mathbf{k}'_i\right) \\ &\times S_n(-\mathbf{k}', \mathbf{k}'_1, \dots, \mathbf{k}'_{n-1}) \left\langle \prod_{i=1}^{n-1} \delta_{L0}(\mathbf{k}'_i) \prod_{j=1}^m \delta(\mathbf{k}_j, \tau_j) \right\rangle. \end{aligned} \quad (47)$$

As described in [9], in the limit of long-wavelength modes the small-scale structures are transported by large-scale perturbations in a uniform fashion. This means that in the limit $k' \rightarrow 0$ for the support of a long-wavelength perturbation $\Delta\delta_{L0}(\mathbf{k}')$, the trajectories of the particles are simply modified as

$$\mathbf{x}(\mathbf{q}, \tau) \rightarrow \mathbf{x}(\mathbf{q}, \tau) + D_+(\tau) \Delta\Psi_{L0}(\mathbf{q}), \quad (48)$$

where \mathbf{q} is the Lagrangian coordinate of the particles and $\Delta\Psi_{L0}(\mathbf{q})$, which is uniform at leading order for $k' \rightarrow 0$,

is the linear displacement field associated with the linear perturbation $\Delta\delta_{L0}$,

$$\Delta\Psi_{L0}(\mathbf{q}) \equiv -\nabla_{\mathbf{q}}^{-1} \cdot \Delta\delta_{L0}. \quad (49)$$

The uniform shift (48) implies that the density field is modified as

$$\delta(\mathbf{x}, \tau) \rightarrow \delta(\mathbf{x} - D_+ \Delta\Psi_{L0}, \tau), \quad (50)$$

which reads in Fourier space (at linear order over $\Delta\Psi_{L0}$) as

$$\delta(\mathbf{k}, \tau) \rightarrow \delta(\mathbf{k}, \tau) - i D_+(\mathbf{k} \cdot \Delta\Psi_{L0}) \delta(\mathbf{k}, \tau). \quad (51)$$

Then, one obtains

$$k' \rightarrow 0: \quad \frac{\mathcal{D}\delta(\mathbf{k})}{\mathcal{D}\delta_{L0}(\mathbf{k}')} = D_+ \frac{\mathbf{k} \cdot \mathbf{k}'}{k'^2} \delta(\mathbf{k}), \quad (52)$$

$$R_\delta^{1,m}(\mathbf{k}') = \left\langle \prod_{j=1}^m \delta(\mathbf{k}_j, \tau_j) \right\rangle \sum_{j=1}^m D_+(\tau_j) \frac{\mathbf{k}_j \cdot \mathbf{k}'}{k'^2}. \quad (53)$$

The results (52)-(53) follow from the weak equivalence principle, that is, from the symmetries of the gravitational dynamics, and are independent of the properties of the density field. Therefore, they remain valid for non-Gaussian initial conditions and Eq.(47) becomes in the squeezed limit, $k' \rightarrow 0$,

$$\begin{aligned} \langle \delta_{L0}(\mathbf{k}') \prod_{j=1}^m \delta(\mathbf{k}_j, \tau_j) \rangle'_{k' \rightarrow 0} &= -P_{\chi_0}(k') \left\langle \prod_{j=1}^m \delta(\mathbf{k}_j, \tau_j) \right\rangle' \\ &\times \sum_{j=1}^m D_+(\tau_j) \frac{\mathbf{k}_j \cdot \mathbf{k}'}{k'^2} + P_{\chi_0}(k') \sum_{n=2}^{\infty} n \int \prod_{i=1}^{n-1} d\mathbf{k}'_i \\ &\times \delta_D\left(\mathbf{k}' - \sum_{i=1}^{n-1} \mathbf{k}'_i\right) S_n(-\mathbf{k}', \mathbf{k}'_1, \dots, \mathbf{k}'_{n-1}) \\ &\times \left\langle \prod_{i=1}^{n-1} \delta_{L0}(\mathbf{k}'_i) \prod_{j=1}^m \delta(\mathbf{k}_j, \tau_j) \right\rangle'. \end{aligned} \quad (54)$$

If the initial conditions obey the squeezed Gaussianity condition (42) the relationship (54) simplifies and takes the same form as in the Gaussian case,

$$\begin{aligned} \langle \delta_{L0}(\mathbf{k}') \prod_{j=1}^m \delta(\mathbf{k}_j, \tau_j) \rangle'_{k' \rightarrow 0} &= -P_{L0}(k') \left\langle \prod_{j=1}^m \delta(\mathbf{k}_j, \tau_j) \right\rangle' \\ &\times \sum_{j=1}^m D_+(\tau_j) \frac{\mathbf{k}_j \cdot \mathbf{k}'}{k'^2}. \end{aligned} \quad (55)$$

Using the fact that on large scales $\delta(\mathbf{k}', \tau') \rightarrow D_+(\tau') \delta_{L0}(\mathbf{k}')$, Eq.(55) also yields

$$\begin{aligned} \langle \delta(\mathbf{k}', \tau') \prod_{j=1}^m \delta(\mathbf{k}_j, \tau_j) \rangle'_{k' \rightarrow 0} &= -P_L(k', \tau') \left\langle \prod_{j=1}^m \delta(\mathbf{k}_j, \tau_j) \right\rangle' \\ &\times \sum_{j=1}^m \frac{D_+(\tau_j)}{D_+(\tau')} \frac{\mathbf{k}_j \cdot \mathbf{k}'}{k'^2}. \end{aligned} \quad (56)$$

Thus, we can see that if the initial conditions show significant non-Gaussianities on large scales and the coefficients (42) do not vanish in the squeezed limit, the consistency relation becomes much more complex than Eq.(56). The $(m+1)$ -squeezed density correlation can no longer be expressed in terms of the m -point small-scale density correlation, as there are additional contributions from all-order mixed correlations (if all coefficients S_n are nonzero). In particular, while the right-hand side in Eq.(56) vanishes at equal times as in the Gaussian case, which means that one must consider subleading contributions, the new terms in Eq.(54) do not vanish.

B. Biased tracers

The consistency relations of the form (56) in the Gaussian case also apply to biased tracers [3–9]. Indeed, as recalled in section IV A, these consistency relations follow from the weak equivalence principle, which states that all matter particles and macroscopic objects fall at the same rate in a gravitational potential. This means that under the almost uniform force $\nabla^{-1} \cdot \Delta \delta_{L0}$, associated with the large-scale perturbation $\Delta \delta_{L0}$, all particles and macroscopic objects experience the uniform shift (48). Small-scale astrophysical processes, such as galaxy and star formation, are not modified by this uniform displacement so that all matter distributions, including galaxy, cluster, and other biased tracers distributions, are simply shifted as in (50). This is somewhat similar to the Galilean invariance of usual hydrodynamical systems, and galaxies form and evolve following the global flow of the system. Here, because of the time-dependent cosmological and gravitational setting, the uniform shift (48) involves the initial condition and the linear growing mode $D_+(\tau)$.

Therefore, the consistency relation (54) writes for biased tracers as

$$\begin{aligned} \langle \delta(\mathbf{k}', \tau') \prod_{j=1}^m \delta_g(\mathbf{k}_j, \tau_j) \rangle'_{k' \rightarrow 0} &= -P_\chi(k', \tau') \langle \prod_{j=1}^m \delta_g(\mathbf{k}_j, \tau_j) \rangle' \\ &\times \sum_{j=1}^m \frac{D_+(\tau_j)}{D_+(\tau')} \frac{\mathbf{k}_j \cdot \mathbf{k}'}{k'^2} + \frac{P_\chi(k', \tau')}{D_+(\tau')} \sum_{n=2}^{\infty} n \int \prod_{i=1}^{n-1} d\mathbf{k}'_i \\ &\times \delta_D \left(\mathbf{k}' - \sum_{i=1}^{n-1} \mathbf{k}'_i \right) S_n(-\mathbf{k}', \mathbf{k}'_1, \dots, \mathbf{k}'_{n-1}) \\ &\times \langle \prod_{i=1}^{n-1} \delta_{L0}(\mathbf{k}'_i) \prod_{j=1}^m \delta_g(\mathbf{k}_j, \tau_j) \rangle', \end{aligned} \quad (57)$$

where δ_g is the density contrast of the biased tracers, such as galaxies. Here we again used the large-scale asymptote, $\delta(\mathbf{k}', \tau') \rightarrow D_+(\tau') \delta_{L0}(\mathbf{k}')$, to replace $\delta_{L0}(\mathbf{k}')$ by $\delta(\mathbf{k}', \tau')$, where τ' is any arbitrary time. Let us emphasize that Eq.(57) is exact and does not make any assumption about the bias of the tracers. In particular, it might be used to check the self-consistency or constrain phenomenological models of biasing.

Again, if the initial conditions obey the squeezed Gaussianity condition (42), the relationship (57) simplifies and takes the same form as in the Gaussian case,

$$\begin{aligned} \langle \delta(\mathbf{k}', \tau') \prod_{j=1}^m \delta_g(\mathbf{k}_j, \tau_j) \rangle'_{k' \rightarrow 0} &= -P(k', \tau') \langle \prod_{j=1}^m \delta_g(\mathbf{k}_j, \tau_j) \rangle' \\ &\times \sum_{j=1}^m \frac{D_+(\tau_j)}{D_+(\tau')} \frac{\mathbf{k}_j \cdot \mathbf{k}'}{k'^2}. \end{aligned} \quad (58)$$

In practice, it could be convenient to write all density factors in the left-hand side of Eq.(58) in terms of the galaxy (or tracers) density field, instead of the mixed matter-galaxy polyspectra. This is only possible if we have a linear deterministic relation between the galaxy and matter density fields on large scales,

$$k' \rightarrow 0 : \quad \delta_g(\mathbf{k}', \tau') = b_1(k', \tau') \delta(\mathbf{k}', \tau'), \quad (59)$$

hence,

$$P_g(k', \tau') \rightarrow b_1(k', \tau')^2 P(k', \tau'), \quad (60)$$

where P_g is the galaxy power spectrum. We do not need to assume that the linear bias b_1 is scale-independent. In fact, it is well-known that for non-Gaussian initial conditions the halo bias can show a significant scale dependence. In particular, for the quadratic local model (29) the deviation of the bias from its Gaussian value behaves as $\Delta b_1 \propto f_{NL}(b_1 - 1)/\alpha(k)$, which diverges as k^{-2} at low k [19]. This is related to the fact that such a model has strong non-Gaussianities on large scales and does not verify the squeezed Gaussianity condition (42). In any case, whenever the asymptote (59) holds, Eq.(58) simplifies (in the case of squeezed Gaussianity) as

$$\begin{aligned} b_1(k', \tau') \langle \delta_g(\mathbf{k}', \tau') \prod_{j=1}^m \delta_g(\mathbf{k}_j, \tau_j) \rangle'_{k' \rightarrow 0} &= -P_g(k', \tau') \\ &\times \langle \prod_{j=1}^m \delta_g(\mathbf{k}_j, \tau_j) \rangle' \sum_{j=1}^m \frac{D_+(\tau_j)}{D_+(\tau')} \frac{\mathbf{k}_j \cdot \mathbf{k}'}{k'^2}. \end{aligned} \quad (61)$$

We can expect the linear relationship (59) to hold for most cases, for many astrophysical tracers and for both Gaussian and non-Gaussian initial conditions. Both the matter and tracer density fluctuations $\delta^{(R)}(\mathbf{x})$ and $\delta_g^{(R)}(\mathbf{x})$, smoothed over a large radius R , are small for sufficiently large R , and Eq.(59) may be interpreted as the first-order term in a Taylor expansion (we can also expect the different non-local terms associated with tidal effects to be small in this large-scale limit).

C. Bispectrum

The simplest density consistency relation is obtained for the squeezed bispectrum, with $m = 2$. We consider the case of biased tracers for the sake of generality, in

the general and exact form (57) that does not explicitly involve the bias and makes no assumption on the biasing of the tracers. The case of the matter density contrast δ is recovered by replacing $\delta_g(\mathbf{k})$ and $P_g(k)$ by $\delta(\mathbf{k})$ and $P(k)$

1. Scenarios with squeezed Gaussianity

For models that satisfy the squeezed Gaussianity criterion (42), Eq.(58) gives the usual result

$$\langle \delta(\mathbf{k}', \tau') \delta_g(\mathbf{k}_1, \tau_1) \delta_g(\mathbf{k}_2, \tau_2) \rangle'_{k' \rightarrow 0} = -P(k', \tau') \times P_g(k_1; \tau_1, \tau_2) \left[\frac{D_+(\tau_1)}{D_+(\tau')} \frac{\mathbf{k}_1 \cdot \mathbf{k}'}{k'^2} + \frac{D_+(\tau_2)}{D_+(\tau')} \frac{\mathbf{k}_2 \cdot \mathbf{k}'}{k'^2} \right]. \quad (62)$$

This takes the same form as the consistency relation obtained for Gaussian initial conditions, but it includes the effect of primordial non-Gaussianities (up to first order over the kernels S_n) as the bispectrum in the left-hand side and the nonlinear power spectrum $P_g(k_1; \tau_1, \tau_2)$ in the right-hand side are sensitive to these primordial non-Gaussianities. As in the Gaussian case, this relation vanishes at equal times, which means that the equal-time bispectrum is governed by the subleading contributions [12, 14].

If the bias is deterministic and linear on large scales, as in Eq.(59), we can replace $\delta(\mathbf{k}')$ and $P(k')$ by $\delta_g(\mathbf{k}')/b_1$ and $P_g(k')/b_1^2$ using Eq.(60), as was done in Eq.(61).

2. S_3 -type primordial non-Gaussianity

In the case of the S_3 -type models introduced in section II C, where only the kernel S_3 is nonzero, the relation (57) gives

$$\begin{aligned} \langle \delta(\mathbf{k}', \tau') \delta_g(\mathbf{k}_1, \tau_1) \delta_g(\mathbf{k}_2, \tau_2) \rangle'_{k' \rightarrow 0} &= -P(k', \tau') \\ &\times P_g(k_1; \tau_1, \tau_2) \left[\frac{D_+(\tau_1)}{D_+(\tau')} \frac{\mathbf{k}_1 \cdot \mathbf{k}'}{k'^2} + \frac{D_+(\tau_2)}{D_+(\tau')} \frac{\mathbf{k}_2 \cdot \mathbf{k}'}{k'^2} \right] \\ &+ 3 \frac{P(k', \tau')}{D_+(\tau')} \int d\mathbf{k}'_1 d\mathbf{k}'_2 \delta_D(\mathbf{k}' - \mathbf{k}'_1 - \mathbf{k}'_2) \\ &\times S_3(-\mathbf{k}', \mathbf{k}'_1, \mathbf{k}'_2) \langle \delta_{L0}(\mathbf{k}'_1) \delta_{L0}(\mathbf{k}'_2) \delta_g(\mathbf{k}_1, \tau_1) \delta_g(\mathbf{k}_2, \tau_2) \rangle'. \end{aligned} \quad (63)$$

Here we used the result (24) that $P_{\chi_0} = P_{L0}$ for these models. As compared with the Gaussian case, there is an additional contribution to the consistency relation involving the mixed four-point correlation.

At equal times, the first term in the right-hand side of Eq.(63) again vanishes. This is because it arises from the uniform displacement of the small-scale structures by the long-wavelength mode k' [9], which cannot be detected from the properties of the equal-time density field. However, this term is only the leading-order factor associated with the gravitational dynamics. It scales as $1/k'$ [multiplied by the prefactor $P(k')$] and there are subleading corrections that remain finite at low k' and do not

vanish at equal times [12]. However, the contributions from these latter terms go to zero in the limit $k' \rightarrow 0$ because of the prefactor $P(k') \rightarrow 0$. Then, if the primordial non-Gaussianities are sufficiently high on large scales, the right-hand side will be dominated by the second term. This applies for instance to the usual local model (27), where $S_3(-\mathbf{k}', \mathbf{k}'_1, \mathbf{k}'_2)$ diverges for $k' \rightarrow 0$. Then, we obtain for such models at equal times,

$$\begin{aligned} \langle \delta(\mathbf{k}') \delta_g(\mathbf{k}_1) \delta_g(\mathbf{k}_2) \rangle'_{k' \rightarrow 0} &= 3 \frac{P(k')}{D_+} \int d\mathbf{k}'_1 d\mathbf{k}'_2 \\ &\times \delta_D(\mathbf{k}' - \mathbf{k}'_1 - \mathbf{k}'_2) S_3(-\mathbf{k}', \mathbf{k}'_1, \mathbf{k}'_2) \\ &\times \langle \delta_{L0}(\mathbf{k}'_1) \delta_{L0}(\mathbf{k}'_2) \delta_g(\mathbf{k}_1) \delta_g(\mathbf{k}_2) \rangle'. \end{aligned} \quad (64)$$

If we split the four-point function in the right-hand side of Eq.(64) over connected and disconnected parts, we obtain

$$\begin{aligned} \langle \delta(\mathbf{k}') \delta_g(\mathbf{k}_1) \delta_g(\mathbf{k}_2) \rangle'_{k' \rightarrow 0} &= 6 \frac{P(k')}{D_+} P_{L0,g}(k_1) P_{L0,g}(k_2) \\ &\times S_3(\mathbf{k}', \mathbf{k}_1, \mathbf{k}_2) + 3 \frac{P(k')}{D_+} \int d\mathbf{k}'_1 d\mathbf{k}'_2 \delta_D(\mathbf{k}' - \mathbf{k}'_1 - \mathbf{k}'_2) \\ &\times S_3(-\mathbf{k}', \mathbf{k}'_1, \mathbf{k}'_2) \langle \delta_{L0}(\mathbf{k}'_1) \delta_{L0}(\mathbf{k}'_2) \delta_g(\mathbf{k}_1) \delta_g(\mathbf{k}_2) \rangle'_c, \end{aligned} \quad (65)$$

where the subscript “c” in the last term $\langle \dots \rangle'_c$ denotes the connected part, that is, the trispectrum, and $P_{L0,g} = \langle \delta_{L0} \delta_g \rangle'$ is the mixed linear-nonlinear power spectrum. On large scales, where at zeroth order over S_3 we have $\delta_L = \chi$ and the trispectrum vanishes, we recover the expression (25) (for unbiased tracers or for the matter density contrast), that is, the primordial bispectrum of the linear density field is set by the initial non-Gaussianity. On smaller scales, where $|\mathbf{k}_1| = |\mathbf{k}_2|$ enters the nonlinear regime, we can see that the first term in the right-hand side of Eq.(65) keeps the same form, except that the linear power spectra $P_{L0}(k_1)$ and $P_{L0}(k_2)$ are replaced by the mixed linear-nonlinear power spectra $P_{L0,g}(k_1)$ and $P_{L0,g}(k_2)$, while the trispectrum contribution no longer vanishes. In practice, this means that it is difficult to use the relationship (65) for analytical or observational purposes, because the mixed trispectrum $\langle \delta_{L1} \delta_{L2} \delta_1 \delta_2 \rangle'$ is not easier to model or compute than the bispectrum $\langle \delta' \delta_1 \delta_2 \rangle'$ and it cannot be directly observed. If the kernel $S_3(-\mathbf{k}', \mathbf{k}'_1, \mathbf{k}'_2)$ in the second term peaks at low values of k'_1 and k'_2 that are still in the linear regime, we can replace the mixed trispectrum by the nonlinear trispectrum, as $\delta_L(\mathbf{k}'_1) \simeq \delta(\mathbf{k}_1)$ and $\delta_L(\mathbf{k}'_2) \simeq \delta(\mathbf{k}_2)$. This trispectrum can in principle be measured, so that Eq.(65) could be compared with observations (or simulations). However, because trispectra are usually very noisy and difficult to measure, it is unlikely that the relationship (65) will provide a competitive method to probe such non-Gaussian scenarios.

Again, if the bias is deterministic and linear on large scales, as in Eq.(59), we can replace $\delta(\mathbf{k}')$ and $P(k')$ by $\delta_g(\mathbf{k}')/b_1$ and $P_g(k')/b_1^2$. However, this is not possible for the factors $P_{L0,g}(k_1)$, $P_{L0,g}(k_2)$, and $\langle \delta_{L0}(\mathbf{k}'_1) \delta_{L0}(\mathbf{k}'_2) \delta_g(\mathbf{k}_1) \delta_g(\mathbf{k}_2) \rangle'_c$, as the wavenumbers k_1, k_2, k'_1 and k'_2 may be in the nonlinear regime where the relationship (59) breaks down.

3. Explicit examples of primordial non-Gaussianity

Let us examine the specific models of primordial non-Gaussianity introduced in Sec. IID to see whether Eqs. (64) or (65) receive a non-vanishing correction or not. This is set by the behavior at low k' of the expression (35). As noticed in section IID 3, for the models defined by the template (30) the expression (35) diverges at low k' , along with the bispectrum in the squeezed limit. The divergence is strongest for the local model (29). Then, the standard consistency relation is largely violated. In general, models with a large squeezed bispectrum lead to a non-vanishing equal-time correlation. Thus, apart from a difficulty in predicting the size of these corrections, Eq. (65) has a sizable effect for primordial non-Gaussianities with a large squeezed bispectrum, and this relation may be used as a consistency check of other non-Gaussian probes.

D. Lack of constraints on equal-times bias models

We may note that for local bias models, where we write expansions such as $\delta_g = b_1\delta + b_2\delta^2/2 + \dots$, the galaxy power spectrum and the matter-galaxy-galaxy bispectrum read up to linear order over b_2 as

$$P_g(k) = b_1^2 P(k) + b_1 b_2 \int d\mathbf{k}'_1 d\mathbf{k}'_2 \delta_D(\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}) \times \langle \delta(\mathbf{k}'_1) \delta(\mathbf{k}'_2) \delta(-\mathbf{k}) \rangle', \quad (66)$$

and

$$\begin{aligned} \langle \delta(\mathbf{k}') \delta_g(\mathbf{k}_1) \delta_g(\mathbf{k}_2) \rangle' &= b_1^2 \langle \delta(\mathbf{k}') \delta(\mathbf{k}_1) \delta(\mathbf{k}_2) \rangle' \\ &+ b_1 b_2 P(k') [P(k_1) + P(k_2)] + \frac{b_1 b_2}{2} \int d\mathbf{k}'_1 d\mathbf{k}'_2 \\ &\times [\delta_D(\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_1) \langle \delta(\mathbf{k}') \delta(\mathbf{k}_2) \delta(\mathbf{k}'_1) \delta(\mathbf{k}'_2) \rangle'_c \\ &+ \delta_D(\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_2) \langle \delta(\mathbf{k}') \delta(\mathbf{k}_1) \delta(\mathbf{k}'_1) \delta(\mathbf{k}'_2) \rangle'_c]. \quad (67) \end{aligned}$$

We may compare with Eq.(62), obtained for Gaussian or squeezed-Gaussianity initial conditions. These two expressions are actually quite different. The expression (67) is usually taken for equal times and it explicitly involves the linear and quadratic bias parameters b_1 and b_2 . In contrast, the relation (62) vanishes at equal times and it does not explicitly involve b_1 and b_2 . Nevertheless, these two relations are not contradictory at equal times. Indeed, the consistency relation (62) only gives the leading-order contribution at unequal times, which is actually much greater than the second term in Eq.(67) by a factor $1/k'$, but does not give the explicit expression of the equal-times bispectrum, although its vanishing at equal times implies that the equal-times bispectrum must grow more slowly than $P_L(k')/k'$ at low k' , which is satisfied by Eq.(67). Therefore, Eq.(67) at equal times does not contradict Eq.(62) and it involves higher-order contributions that are not taken into account in this consistency relation.

Ref. [7] explicitly checked that a quadratic local bias model satisfies the consistency relation for the galaxy bispectrum, up to one-loop order. In fact, we can see that any bias model, where the galaxy density field is a functional of the same-time matter density field, automatically satisfies the consistency relations (57), for both Gaussian and non-Gaussian initial conditions and at all orders. Indeed, let us write the galaxy density field as an expansion over the matter density field,

$$\delta_g(\mathbf{k}, \tau) = \sum_{\ell=0}^{\infty} \int d\mathbf{k}_1 \dots d\mathbf{k}_\ell \delta_D(\mathbf{k}_1 + \dots + \mathbf{k}_\ell - \mathbf{k}) \times b_\ell(\mathbf{k}_1, \dots, \mathbf{k}_\ell; \tau) \delta(\mathbf{k}_1, \tau) \dots \delta(\mathbf{k}_\ell, \tau). \quad (68)$$

This expression goes beyond the local bias models, because the \mathbf{k} -dependence of the kernels b_ℓ allows us to include tidal terms, which are naturally generated by the dynamics [26–28]. We can also include stochasticity as the kernels b_ℓ can be stochastic, so that after the average $\langle \dots \rangle$ over the initial conditions we perform a second average over some stochastic variables ϵ_i that are uncorrelated with the density field. Thus, Eq.(68) is a very general bias model, which only assumes that the galaxy density field can be expanded over powers of the same-time matter density field. Then, substituting Eq.(68) into the left-hand side in Eq.(57) and using the matter consistency relation (54), we recover the right-hand side in Eq.(57). This is straightforward for the non-Gaussian S_n -dependent term in Eq.(57), and it also applies to the first standard term because each factor of the form $(\mathbf{k}_1^{(j)} + \dots + \mathbf{k}_{\ell_j}^{(j)}) \cdot \mathbf{k}'/k'^2$, associated with the expansion (68) of a factor $\delta_g(\mathbf{k}_j)$, simplifies as $\mathbf{k}_j \cdot \mathbf{k}'/k'^2$ thanks to the Dirac factors in Eq.(68). Therefore, we find that the consistency relations (57) do not provide very useful constraints or guidelines for the building of analytic biasing models, as they do not give any information on the kernels b_ℓ and all bias models (68) satisfy these consistency relations. On the other hand, this ensures that the general bias models (68) do not face unphysical inconsistencies at this level.

As noticed in Ref. [7], this result is expected because the bias model (68) verifies the symmetries of the system. More explicitly, the galaxy density field defined by Eq.(68) is transported in a uniform fashion by a large-scale mode, exactly as in Eq.(50) for the matter density field, as all factors $\delta(\mathbf{x}_i, \tau)$ (working in configuration space) are shifted by the same uniform displacement $D_+(\tau)\Delta\Psi_{L0}$ as in Eq.(48). This implies that δ_g also verifies the response (52) to a large-scale perturbations, which directly leads to the consistency relations.

On the other hand, if we consider a bias model that involves unequal-times matter density fields $\delta(\mathbf{k}_i, \tau_i)$ in Eq.(68), with kernels $b_\ell(\mathbf{k}_1, \tau_1; \dots; \mathbf{k}_\ell, \tau_\ell)$ and integrals over the past times τ_i , the consistency relations are generically violated. Indeed, the different factors $\delta(\mathbf{x}_i, \tau_i)$ (working again in configuration space) are shifted by the different uniform displacements $D_+(\tau_i)\Delta\Psi_{L0}$. In Fourier space, we can no longer use the simplification

$(\mathbf{k}_1^{(j)} + \dots + \mathbf{k}_{\ell_j}^{(j)}) \cdot \mathbf{k}'/k'^2 = \mathbf{k}_j \cdot \mathbf{k}'/k'^2$, because each factor $\mathbf{k}_i^{(j)} \cdot \mathbf{k}'/k'^2$ is multiplied by a different time-dependent factor $D_+(\tau_i^{(j)})$. This means that the consistency relations provide strong constraints on bias models that write the galaxy density field as a functional of different-times matter density fields. However, in practice bias models have the equal-time form (68), to avoid unnecessarily complex models that display too many free parameters and free functions, and to focus on the simplest models.

V. CONSISTENCY RELATIONS FOR VELOCITY AND MOMENTUM FIELDS

As we recalled in the previous section, the leading-order effect of a long wavelength perturbation is to move smaller structures by uniform shift, which leads to the functional derivative (52). Then, if we consider equal-time statistics of the density field, in the Gaussian case we cannot see any effect (as we cannot detect a uniform shift by such probes) and the sum in the right-hand side of Eq.(55) vanishes (using $\sum \mathbf{k}_j = 0$). On the other hand, for scenarios with high primordial non-Gaussianities on large scales the new term in the right-hand side of Eq.(54) remains significant in the squeezed limit and we obtain a nonzero relationship, as in Eq.(65).

As pointed out in [15], in the Gaussian case we can obtain nontrivial consistency relations by cross correlating density and velocity, or momentum, fields. Indeed, the uniform displacement of the small-scale structures also leads to a modification of the amplitude of the local velocity and the latter can be detected at equal times by measuring the velocity or momentum field. In a fashion similar to the procedure described in section IV A for density correlations, we can obtain these consistency relations from the general relationship (40) by taking the quantities $\{\rho_i\}$ to be a combination of density and velocity fields. In this paper we only recall the expressions that were obtained for the Gaussian case, the derivation can be found in [15].

For the velocity field $\mathbf{v}(\mathbf{x}, \tau)$ we obtain the consistency relation

$$\begin{aligned} \langle \delta(\mathbf{k}', \tau') \prod_{j=1}^m \delta(\mathbf{k}_j, \tau_j) \prod_{j=m+1}^{m+\ell} \mathbf{v}(\mathbf{k}_j, \tau_j) \rangle'_{k' \rightarrow 0} &= -P_L(k', \tau') \\ &\times \left\{ \left\langle \prod_{j=1}^m \delta(\mathbf{k}_j, \tau_j) \prod_{j=m+1}^{m+\ell} \mathbf{v}(\mathbf{k}_j, \tau_j) \right\rangle' \sum_{i=1}^{m+\ell} \frac{D_+(\tau_i)}{D_+(\tau')} \frac{\mathbf{k}_i \cdot \mathbf{k}'}{k'^2} \right. \\ &+ \sum_{i=m+1}^{m+\ell} \left\langle \prod_{j=1}^m \tilde{\delta}(\mathbf{k}_j, \tau_j) \prod_{j=m+1}^{i-1} \mathbf{v}(\mathbf{k}_j, \tau_j) \times \right. \\ &\left. \left(\frac{(dD_+/d\tau)(\tau_i)}{D_+(\tau')} \mathbf{i} \frac{\mathbf{k}'}{k'^2} \delta_D(\mathbf{k}_i) \right) \prod_{j=i+1}^{m+\ell} \mathbf{v}(\mathbf{k}_j, \tau_j) \right\rangle' \left. \right\}. \end{aligned} \quad (69)$$

The first term takes the same form as the relation (56) for the density contrast, while the second term with the

Dirac factor $\delta_D(\mathbf{k}_i)$ is new. The first term again corresponds to the uniform translation of the fields by the large-scale modes, as in Eqs.(50)-(51). The new second term corresponds to the fact that the amplitude of the velocity field is also modified by a uniform amount.

The last term in Eq.(69) has the advantage that it does not vanish at equal times. However, because it comes with a Dirac factor $\delta_D(\mathbf{k}_i)$ it vanishes in Fourier space for $\mathbf{k}_i \neq 0$. As noticed in [15], a simple way to make this term relevant is to consider composite operators, that is, products of the velocity field with other fields. This leads to convolutions in Fourier space that will probe this term for all wave numbers \mathbf{k} of the composite field. Thus, we define the momentum \mathbf{p} as

$$\mathbf{p} = (1 + \delta)\mathbf{v}, \quad (70)$$

and we obtain the consistency relations

$$\begin{aligned} \langle \delta(\mathbf{k}', \tau') \prod_{j=1}^m \delta(\mathbf{k}_j, \tau_j) \prod_{j=m+1}^{m+\ell} \mathbf{p}(\mathbf{k}_j, \tau_j) \rangle'_{k' \rightarrow 0} &= -P_L(k', \tau') \\ &\times \left\{ \left\langle \prod_{j=1}^m \delta(\mathbf{k}_j, \tau_j) \prod_{j=m+1}^{m+\ell} \mathbf{p}(\mathbf{k}_j, \tau_j) \right\rangle' \sum_{i=1}^{n+m} \frac{D_+(\tau_i)}{D_+(\tau')} \frac{\mathbf{k}_i \cdot \mathbf{k}'}{k'^2} \right. \\ &+ \sum_{i=m+1}^{m+\ell} \frac{(dD_+/d\tau)(\tau_i)}{D_+(\tau')} \left\langle \prod_{j=1}^m \delta(\mathbf{k}_j, \tau_j) \prod_{j=m+1}^{i-1} \mathbf{p}(\mathbf{k}_j, \tau_j) \right. \\ &\left. \times \left(\mathbf{i} \frac{\mathbf{k}'}{k'^2} [\delta_D(\mathbf{k}_i) + \delta(\mathbf{k}_i, \tau_i)] \right) \prod_{j=i+1}^{m+\ell} \mathbf{p}(\mathbf{k}_j, \tau_j) \right\rangle' \left. \right\} \end{aligned} \quad (71)$$

In contrast with the consistency relation (56) obtained for the density field, the right-hand side in Eq.(71) does not vanish at equal times, thanks to the new terms associated with the change of the amplitude of the velocity. This leads to a nontrivial consistency relation at equal times, when $\tau' = \tau_1 = \dots = \tau_{m+\ell}$,

$$\begin{aligned} \langle \delta(\mathbf{k}') \prod_{j=1}^m \delta(\mathbf{k}_j) \prod_{j=m+1}^{m+\ell} \mathbf{p}(\mathbf{k}_j) \rangle'_{k' \rightarrow 0} &= -i P_L(k') \frac{d \ln D_+}{d\tau} \\ &\times \sum_{i=m+1}^{m+\ell} \left\langle \prod_{j=1}^m \delta(\mathbf{k}_j) \prod_{j=m+1}^{i-1} \mathbf{p}(\mathbf{k}_j) \left(\frac{\mathbf{k}'}{k'^2} [\delta_D(\mathbf{k}_i) + \delta(\mathbf{k}_i)] \right) \right. \\ &\left. \times \prod_{j=i+1}^{m+\ell} \mathbf{p}(\mathbf{k}_j) \right\rangle' \end{aligned} \quad (72)$$

where we did not write the common time τ of all fields. We can also obtain a consistency relation that involves both the density and velocity fields δ and \mathbf{v} , together with the momentum field \mathbf{p} , following the same approach.

To obtain scalar consistency relations, instead of the vector quantities (69) and (71), we consider the divergence of the momentum field,

$$\lambda(\mathbf{x}, \tau) \equiv \nabla_{\mathbf{x}} \cdot [(1 + \delta)\mathbf{v}], \quad \lambda(\mathbf{k}, \tau) \equiv \mathbf{i} \mathbf{k} \cdot \mathbf{p}(\mathbf{k}, \tau). \quad (73)$$

Then, the consistency relation for the divergence λ follows from Eq.(71). This gives

$$\begin{aligned} \langle \delta(\mathbf{k}', \tau') \prod_{j=1}^m \delta(\mathbf{k}_j, \tau_j) \prod_{j=m+1}^{m+\ell} \lambda(\mathbf{k}_j, \tau_j) \rangle'_{k' \rightarrow 0} &= -P_L(k', \tau') \\ &\times \left\{ \langle \prod_{j=1}^m \delta(\mathbf{k}_j, \tau_j) \prod_{j=m+1}^{m+\ell} \lambda(\mathbf{k}_j, \tau_j) \rangle' \sum_{i=1}^{m+\ell} \frac{D_+(\tau_i)}{D_+(\tau')} \frac{\mathbf{k}_i \cdot \mathbf{k}'}{k'^2} \right. \\ &- \sum_{i=m+1}^{m+\ell} \langle \delta(\mathbf{k}_i, \tau_i) \prod_{j=1}^m \delta(\mathbf{k}_j, \tau_j) \prod_{\substack{j=m+1 \\ j \neq i}}^{m+\ell} \lambda(\mathbf{k}_j, \tau_j) \rangle' \\ &\left. \times \frac{(dD_+/d\tau)(\tau_i)}{D_+(\tau')} \frac{\mathbf{k}_i \cdot \mathbf{k}'}{k'^2} \right\}. \end{aligned} \quad (74)$$

At equal times this simplifies as

$$\begin{aligned} \langle \delta(\mathbf{k}') \prod_{j=1}^m \delta(\mathbf{k}_j) \prod_{j=m+1}^{m+\ell} \lambda(\mathbf{k}_j) \rangle'_{k' \rightarrow 0} &= P_L(k') \frac{d \ln D_+}{d\tau} \\ &\times \sum_{i=m+1}^{m+\ell} \frac{\mathbf{k}_i \cdot \mathbf{k}'}{k'^2} \langle \delta(\mathbf{k}_i) \prod_{j=1}^m \delta(\mathbf{k}_j) \prod_{\substack{j=m+1 \\ j \neq i}}^{m+\ell} \lambda(\mathbf{k}_j) \rangle'. \end{aligned} \quad (75)$$

The simplest consistency relation that does not vanish at equal times is the equal-time bispectrum with one momentum field. From Eqs.(72) and (75) we obtain

$$\langle \delta(\mathbf{k}') \delta(\mathbf{k}) \mathbf{p}(-\mathbf{k}) \rangle'_{k' \rightarrow 0} = -i \frac{\mathbf{k}'}{k'^2} \frac{d \ln D_+}{d\tau} P_L(k') P(k) \quad (76)$$

and

$$\langle \delta(\mathbf{k}') \delta(\mathbf{k}) \lambda(-\mathbf{k}) \rangle'_{k' \rightarrow 0} = - \frac{\mathbf{k} \cdot \mathbf{k}'}{k'^2} \frac{d \ln D_+}{d\tau} P_L(k') P(k). \quad (77)$$

Here $P(k)$ is the nonlinear density power spectrum and these relations remain valid in the nonperturbative nonlinear regime.

The consistency relations (69)-(77) were derived in [15] for the Gaussian case. As for the density contrast studied in section IV, for non-Gaussian models these consistency relations are modified by the additional terms that arise from the S_n -dependent factors in Eq.(40), in a fashion similar to Eq.(54). We do not explicitly write these terms here, because they lead to consistency relations that are probably of little practical value since they involve high-order mixed correlations $\langle \prod \delta_L \prod \delta \prod \mathbf{p} \rangle'$ or $\langle \prod \delta_L \prod \delta \prod \mathbf{p} \rangle'$ that are difficult to measure or predict. On the other hand, for non-Gaussian scenarios that obey the squeezed Gaussianity conditions (42) the consistency relations (69)-(77) remain valid, as Eq.(56) for the density contrast. Again, even if these relations now take the same form as in the Gaussian case, they go beyond the Gaussian result because the correlation functions in both sides of these relations depend on the properties of the initial conditions, hence on the kernels S_n .

VI. CONCLUSIONS

In this paper we have described how the consistency relations of cosmological large-scale structures are modified when the initial conditions are not Gaussian. We consider very general scenarios, where the primordial density field can be written as a nonlinear functional of a Gaussian field χ_0 , up to all orders over χ_0 , or more generally, where the probability distribution of the primordial density field can be expanded around the Gaussian, up to all orders over δ_{L0} . We also give the relationship between these two formalisms. As the primordial non-Gaussianities should be small, to remain consistent with observations, we work at linear order over the non-Gaussianity kernels $f_{\text{NL}}^{(n)}$ or S_n . We give the constraints that must be verified by these kernels, which arise from the normalization conditions $\langle 1 \rangle = 1$ and $\langle \delta_{L0} \rangle = 0$. A simple example is provided by the S_3 -type primordial non-Gaussianity, which is fully defined by the power spectrum and bispectrum.

We show how the approach used for the Gaussian case applies to these non-Gaussian scenarios. We can still obtain a relationship between correlation and response functions, but it is generally much more intricate as it involves all-order mixed correlations such as $\langle \prod \delta_L \prod \delta \rangle'$. For scenarios that converge to the Gaussian in the squeezed limit, we recover the simple relationship obtained in the Gaussian case, even though the small-scale modes may be strongly affected by the primordial non-Gaussianity. We give the explicit conditions for this simplification to hold.

Then, as for the Gaussian case, we use this general relationship to derive the consistency relations for density, velocity, and momentum fields, as well as for biased tracers. We discuss in more details the relation obtained for the density bispectrum, especially for the case of the S_3 -type primordial non-Gaussianity. We describe the form it takes at equal times, when the Gaussian-like term vanishes and we are dominated by the new contributions associated with the primordial non-Gaussianity. Unfortunately, this expression involves a complicated mixed trispectrum $\langle \delta_L \delta_L \delta \delta \rangle'$ and it may not be very practical. In the case of scenarios with squeezed Gaussianity, we briefly discuss the relations obtained for the bispectrum with one momentum field, as they remain nonzero at equal times.

We find that, for both Gaussian and non-Gaussian initial conditions, the consistency relations are automatically satisfied by very general bias models, where the galaxy density field can be written as an expansion over powers of the same-time density field (including nonlocal terms such as tidal fields and stochasticity). Thus, these consistency relations do not provide any information on the generalized bias kernels $b_\ell(\mathbf{k}_1, \dots, \mathbf{k}_\ell; \tau)$. On the other hand, this means that such bias models do not face unphysical inconsistencies at this level. In contrast, the consistency relations would constrain bias models that write the galaxy density field as a functional of different-times

density fields, but such models are not used in practice.

To conclude, we find that the usual consistency relations, that were derived for Gaussian initial conditions, remain valid for a large class of primordial non-Gaussianities. Even though the small-scale modes probed by these relations may be highly nonlinear or strongly affected by the primordial non-Gaussianity, it is sufficient for their validity that primordial non-Gaussianities vanish in the squeezed limit, when one mode is pushed to large scales. Therefore, these consistency relations cannot be used as a precise test of such scenarios, in the sense that they do not discriminate between the Gaussian case and these models. On the other hand, if the primordial non-Gaussianities remain large in the squeezed limit, the consistency relations are modified and involve additional mixed linear-nonlinear correlations $\langle \prod \delta_L \prod \delta \rangle'$. Therefore, cosmological consistency

relations can be used as a test of squeezed primordial non-Gaussianities. A lack of deviation from the Gaussian-case prediction would constrain the amplitude of squeezed primordial non-Gaussianities, whereas a measured deviation would rule out both the Gaussian scenario and the non-Gaussian models with vanishing squeezed primordial non-Gaussianities.

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